EXPERIMENTAL VALIDATION OF TWO NONLINEAR IDENTIFICATION TECHNIQUES ON A SINGLE DEGREE OF FREEDOM SYSTEM

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ABSTRACT

The identification of a single degree of freedom non-linear system is performed on experimental data using two different techniques : a method based on the Wavelet transform and a Direct Parameter estimation method. Both techniques are based on the analysis of the system free responses and result in the estimation of linear and non-linear physical parameters. The agreement between the two methods was found to be encouraging and opens to the possibility of extending them to the analysis of more complicated systems.

1. INTRODUCTION

The aim of this paper is to compare two different identification techniques : a method based on the Wavelet transform (WT) and a Direct Parameter estimation method. The comparison is performed on the example of a non-linear single degree of freedom (SDOF) system using free response measurement data. The first method (WT) is based on the detection of the decay envelope which is fitted to its theoretical formulation using non-linear averaging techniques. The Direct Parameter estimation technique expresses the restoring force as a mathematical function of the displacement and of the velocity by least square fitting.

2. THEORETICAL BACKGROUND

2.1 The wavelet Transform for nonlinear identification

In this work an attempt is made to extend the WT analysis to the free vibration of non-linear SDOF systems. The WT may be considered as an improved tool for envelope and phase estimation [1] with respect to the Hilbert Transform (HT) which has been extensively used for the identification of non-linear systems ([3],[4]).

A very general formulation for the damping force acting on the system is considered here, the dissipation mechanism being modelled as a *n*-th order power series of the velocity. The analytical expression of the free response of systems with this kind of damping has been derived using first order approximation methods, i.e. the Method of Multiple Scales and the Method of Averaging [5]. The presence of non-linear stiffness can also be considered : the analytical derivation for the free response has been derived through the methods mentioned above for the case of stiffness of the generalised odd order *m*.

2.1.1 Response of a nonlinear single degree of freedom system

The equation of motion of a weakly nonlinear SDOF system can be written with the well known expression:

$$\ddot{u}(t) + \omega_0^2 \cdot u(t) = \varepsilon \cdot f(u,\dot{u}) \tag{2.1}$$

where $\omega_0 = \sqrt{k/m}$ is the natural angular frequency of the linear system, ε is a small dimensionless parameter and $f(u,\dot{u})$ a general nonlinear function of u, $\dot{u}$. With the assumption of ε being small, a number of approximate methods can be used for the determination of the analytical solution of equation (2.1) (see Nayfeh and Mook [5]). In particular, using the method of averaging [5], a solution can be sought in the form:

$$u(t) = k(t) \cdot \cos\big(\omega_0 \cdot t + \gamma(t)\big) = k(t) \cdot \cos\phi(t) \tag{2.2}$$

where $k(t)$ and $\gamma(t)$ are the amplitude and the phase modulation of the system free response respectively.
For small ε, it can be stated that the variations of $k(t)$ and $\gamma(t)$ over one period of $\cos\phi$ are negligible when compared to the variations of $\phi(t)$. By averaging over one period, the expressions describing the slow variations of $k(t)$ and $\gamma(t)$ can be obtained :

$$\begin{aligned} \dot{k} &= -\frac{\varepsilon}{2\pi \cdot \omega_0} \cdot \int_0^{2\pi} \sin\phi \cdot f(k \cdot \cos\phi, -\omega_0 \cdot k \cdot \sin\phi)\, d\phi \\ \dot{\gamma} &= -\frac{\varepsilon}{2\pi \cdot k \cdot \omega_0} \cdot \int_0^{2\pi} \cos\phi \cdot f(k \cdot \cos\phi, -\omega_0 \cdot k \cdot \sin\phi)\, d\phi \end{aligned} \tag{2.3}$$

Equations (2.3) and the Method of Averaging in general allow to obtain an approximate analytical expression describing the free behaviour of a SDOF system, for different forms of the function $f(u,\dot{u})$.
This work focus on the study of systems with combined generalised nonlinear damping mechanisms, for which the function $f(u,\dot{u})$ has the following form :

$$\varepsilon \cdot f(u,\dot{u}) = \varepsilon \cdot f(\dot{u}) = -\sum_{i=0}^{n} \mu_i \cdot |\dot{u}|^i \cdot sign(\dot{u}) \tag{2.4}$$

where n is the maximum order considered in the damping mechanism and μ_i is the *i*-th damping coefficient, normalised to the mass of the system.
Equation (2.4) allows to consider the simultaneous effect of different simple damping mechanisms acting on the system, i.e. Coulomb damping, viscous (linear)

damping, aerodynamic forces (quadratic damping), etc... In particular equation (2.4) gives the opportunity to approximate with a polynomial function the restoring force given by a real dissipation mechanism.
An approximate expression of the free response of a SDOF system with a combined damping mechanism of the generalised order n can still be found from equations (2.3) obtaining :

$$\dot{k} = -\sum_{i=0}^{n} \frac{\varepsilon}{\sqrt{\pi}} \cdot \mu_i \cdot \omega_0^{i-1} \cdot \frac{\Gamma(i/2+1)}{\Gamma(i/2+3/2)} \cdot k^i = -\sum_{i=0}^{n} c_i \cdot k^i \tag{2.5}$$

and

$$\dot{\gamma} = \frac{\varepsilon}{\pi \cdot \omega_0 \cdot k} \cdot \int_0^{\pi} \cos\phi \cdot \sum_{i=0}^{n} \mu_i \cdot \omega_0{}^i \cdot k^i (\sin\phi)^i \cdot d\phi = 0 \tag{2.6}$$

Equation (2.5) is a first order differential equation, whose analytical solution for a generalised value of n is trivial. In this work, the theoretical envelope of a system with combined damping has been determined by numerical integration of equation (2.5).

2.1.2 Identification of the envelope and instantaneous frequency of a signal

The Wavelet Transform (WT) of a signal $x(t)$ is an example of a time-scale decomposition obtained by dilating and translating along the time axis a chosen analysing function named 'wavelet'. The continuous WT is defined as follows :

$$W_g(a,b) = \frac{1}{\sqrt{a}} \cdot \int_{-\infty}^{+\infty} x(t) \cdot g^*\left(\frac{t-b}{a}\right) dt \tag{2.7}$$

where b is the parameter localising the wavelet function in the time domain, a is the dilation parameter defining the analysing window stretching and g^* is the complex conjugate of the basic wavelet function. The wavelet function used in this work is the Morlet's wavelet. It can be shown [2] how the WT gives a time-frequency representation of the signal performing a linear transformation.
The WT has already been used for modal parameters identification in [1]. In particular, it has been highlighted how the WT of signals expressed by equation (2.2) is :

$$W_g(a,b) = \sqrt{a} \cdot k(t) \cdot e^{-(a\cdot\dot{\phi}(t)-\omega_w)^2} \cdot e^{j\phi(t)} \tag{2.8}$$

where $k(t)$ and $\phi(t)$ are generally varying in time. For a fixed frequency value, the modulus of equation (2.8) is directly proportional to the envelope of the signal $k(t)$ while the time derivative of the phase of $W_g(a,b)$ gives the instantaneous frequency content of the signal.

2.1.3 Estimation of the damping coefficients: least square solution

The theoretical solution for a combined damping mechanism defined by equation (2.4), together with the estimation procedure based on the WT has been used for the identification of the damping coefficients μ_i in equation (2.4). Equation (2.5) gives the amplitude of the free response of a system with a known damping mechanism, i.e. with known coefficients μ_i (*i=0,..,n*).

If an identification procedure has to be performed, the damping coefficients μ_i obviously represent the unknowns of the problem, while the amplitude of the decay can be rather easily estimated using the WT (equation (2.8)). If *k(t)* in equation (2.8) is differentiated with respect to time *t* and if expression (2.5) is used, an equation in the *n* unknowns c_i , directly proportional to μ_i is obtained :

$$\dot{k}(t)-\sum_{i=0}^{n} c_i \cdot k^i(t)=0 \tag{2.9}$$

where *k(t)* is the envelope estimated by the WT.

To reduce the influence of noise present in the estimation of *k(t),* it is preferable to integrate equation (2.9) with respect to time:

$$\int_{t_0}^{t} \dot{k}\cdot d\tau-\sum_{i=0}^{n}(c_i\int_{t_0}^{t} k^i\cdot d\tau)=0 \tag{2.10}$$

where t_m-t_0 represents the time duration of the analysed signal.

Let us write for the sake of simplicity :

$$x(t)=\int_{t_0}^{t}\dot{k}(\tau)\cdot d\tau=k(t)-k(t_0); \qquad y_i(t)=\int_{t_0}^{t}k^i(\tau)\cdot d\tau \tag{2.11}$$

the n coefficients c_i can be easily obtained in closed form by imposing the stationarity of the following error function:

$$e=\sum_{j=1}^{m}[x(t_j)-\sum_{i=0}^{n}c_i\cdot y_i(t_j)]^2=0 \tag{2.12}$$

Hence a linear system is obtained with *n* equations in the *n* unknowns c_i. Once the *n* coefficients c_i have been found, the damping coefficients μ_i can be easily calculated from equation (2.5).

2.2 The Direct Parameter Estimation (DPE) method

The DPE method is strictly related to the Restoring Force Method, widely used in the past years ([6] - [9]). It is based on Newton’s second law, which for a SDOF system is:

$$m\cdot\ddot{u}(t)+f(u,\dot{u})=x(t) \tag{2.13}$$

where m is the mass of the system, $x(t)$ is the external force applied to the mass and $f(u,\dot{u})$ is the internal displacement and velocity dependent force, known as 'restoring force'. If the external force $x(t)$ and the acceleration are sampled simultaneously at regular intervals, the value of the restoring force at each sampling instant can be calculated from :

$$f_i = x(t_i) - m \cdot \ddot{u}(t_i) \tag{2.14}$$

where $x(t_i)$ is the i-th sampled value of the input force. If the displacements and the velocities, obtained by integrating the measured acceleration data, are estimated at each sampling instant, a triplet of values $u_i, \dot{u}_i, f_i$ can be obtained. Each triplet specifies a point in the phase plane $(u, \dot{u})$ and the corresponding amplitude of the restoring force. Using this data, a continuous representation of the force surface can be constructed, which gives an easily understandable representation of the system non-linearity. A mathematical model may be obtained by fitting of the data triplets :

$$\sum_{r}^{n_1} \mu_r \cdot \dot{u}_i^{\,r} + \sum_{j}^{n_2} \chi_j \cdot u_i^{\,j} = x_i / m - \ddot{u}_i \qquad (i = 1, \dots, p) \tag{2.15}$$

This equation can be assembled into a matrix relation by considering all the p recorded samples :

$$\begin{bmatrix} u_1 & u^{n_2}{}_1 & \dot{u}_1 & \dot{u}^{n_1}{}_1 \\ u_2 & u^{n_2}{}_2 & \dot{u}_2 & \dot{u}^{n_1}{}_2 \\ \dots & \dots \; \dots & \dots \; \dots & \dots \\ u_p & u^{n_2}{}_p & \dot{u}_p & \dot{u}^{n_1}{}_p \end{bmatrix} \cdot \begin{bmatrix} \chi_1 \\ \dots \\ \chi_{n_2} \\ \mu_1 \\ \dots \\ \mu_{n_1} \end{bmatrix} = \begin{bmatrix} x_1/m - \ddot{u}_1 \\ x_2/m - \ddot{u}_2 \\ \\ x_p/m - \ddot{u}_p \end{bmatrix} \tag{2.16}$$

which may be written in the form :

$$\boldsymbol{A} \cdot \boldsymbol{c} = \boldsymbol{B} \tag{2.17}$$

The vector $\boldsymbol{c}$ of unknown parameters may be determined using the pseudo inverse method or the Singular Value Decomposition (SVD) method.
The normalised mean-squared error (MSE), which is a measure of the accuracy of the fit, is defined as

$$MSE(f) = \frac{100}{N.\sigma_f^2} \sum_{i=1}^{N} (f_i - f_i^{'})^2 \tag{2.18}$$

Where f_i is an array of measured time data and $f_i^{'}$ is the predicted value of f_i.

In practice, the orders n_1 and n_2 used in equation (2.16) to fit the data are not known *a priori*. In order to have an indication on what terms should be included in the summations in equation (2.15), or better, in order to have some means of determining which of the possible terms are significant and which can be safely discarded, the "significance factor" introduced in [9] can be used. Each term of the model $\vartheta(t)$, for example $\vartheta(t) = \chi_3 \cdot u^3$, may be used independently to generate a time-series which will have variance σ_ϑ^2. The significance factor is then defined as :

$$s_\vartheta = 100 \cdot \frac{\sigma_\vartheta^2}{\sigma_{tot}^2} \tag{2.19}$$

where σ_{tot}^2 is the variance of the term $x/m - \ddot{u}$, including all the model terms. Roughly speaking, s_ϑ represents the percentage of contribution to the model variance of the term $\vartheta(t)$ of the restoring force. After the estimation of the parameters, the significance factors are determined and all the terms contributing less than a threshold value s_{min} may be discarded.

This method may be applied using different types of excitation (random, sinusoidal, sweep sine, etc). In the present study, the DPE method was applied to the free responses of the system, i.e. by setting $x(t)=0$ in equations (2.14) through (2.16). In this particular case, the application of the method requires the previous estimation of the mass of the system as all the coefficients in equations (2.15) and (2.16) are mass normalised.

Note that the previous knowledge of the mass is also required for the method based on the Wavelet transform.

3. EXPERIMENTAL VALIDATION OF THE TWO NON-LINEAR IDENTIFICATION TECHNIQUES

The two techniques presented previously have been applied to the measured free responses of a mass suspended by four springs in a way that its dynamic behaviour can be easily assumed to be that of a single degree of freedom system.

The results obtained by the Wavelet Transform (WT) based technique and the Direct Parameter Estimation (DPE) method have been directly compared.

3.1 Experimental set-up

The WT technique requires only to measure the displacements of the system, while the DPE method needs to determine displacements, velocities and accelerations. Considering that velocities and displacements may be obtained, with some care, by integration of the acceleration responses, the DPE method requires to measure the accelerations of the system. Note that this procedure leads to good results, provided the low frequency components, always present in the measured and integrated signals, are previously filtered out. In order to compare the results from the two methods, both displacements and accelerations were measured.

The experimental system is composed of a rectangular plate suspended over a massive base assuring the stability of the system. The plate is linked with the base by four springs and is constrained to move in the vertical direction by four linear supports, hence reproducing, as closely as possible, the behaviour of a spring-mass single degree of freedom system.

The suspended mass including the sensors and the mounting is equal to *21.36 kg* and the global stiffness of the system in the vertical direction is evaluated to *12625 N/m* from a static test. The mass is coupled with the linear guides by four linear bushes with circulating balls, with the aim of reducing the friction between the moving parts. This type of dissipation is not the only one that can be expected in the system, since also structural damping is likely to be present. The decay response of the system may be presumed to be caused by a combination of a linear term, due to friction damping, and of an exponential term, due to structural damping. The simultaneous presence of these effects makes the system non-linear. The aim of the identification procedure is to verify this presumption and to detect other possible forms of dissipation mechanism and/or non-linearity.

3.2 Test description

After imposing an initial displacement to the mass, the free decay responses were obtained by release of the system. Different initial displacement values were first considered, so that the estimation procedures could be tested in different conditions, namely with the system undergoing different numbers of oscillations before returning at rest. Then, some time histories were recorded simply by pushing the mass downwards or upwards, thus simulating an imposed initial velocity. The estimation procedures were carried out for all the test configurations in order to check the repetitiveness and the stability of the identification methods.
The data were sampled at 256 Hz for an acquisition time of 8 seconds, corresponding to 2048 samples, that is about the time required for the system to return at rest.

3.3 Identification of the physical parameters

First, the order of the damping mechanism has been evaluated by estimating the mean squared error and by using the significance factor introduced within the DPE method (equation (2.19)).

The estimated parameters for the two methods are listed in Table 3.1.

For all the test configurations, a first order damping mechanism, i.e. including Coulomb damping and linear damping, has been estimated, both by means of the squared error and of the significance factor. The presumptions briefly discussed in section 3.1 have been therefore confirmed by the identification procedures.

The behaviour of the springs, within the range of displacements considered in the tests, has been identified to be approximately linear. This is also confirmed by the fact that the instantaneous frequency of all the signals remains approximately constant for the duration of the responses.

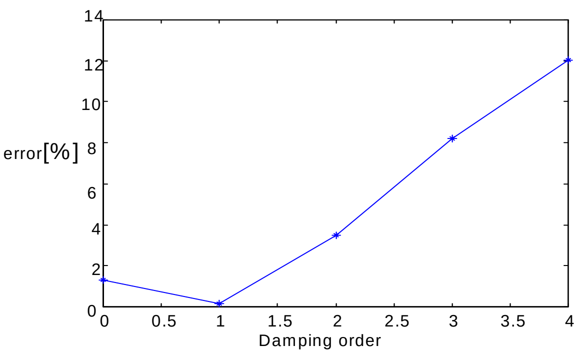
The decay rate is rather low and the part of the signal useful for fitting the model is correspondingly long : in this case, the application of the identification procedures is not too difficult.

In the case of the WT method, the value of the linear stiffness K has been evaluated by calculating the mean value of the instantaneous frequency and by using the estimated value of the suspended mass given in section 2.2 ($K = m \cdot \omega_0^2$). In both methods, the resulting value of the dynamic stiffness K is slightly different than the one estimated from the static test.

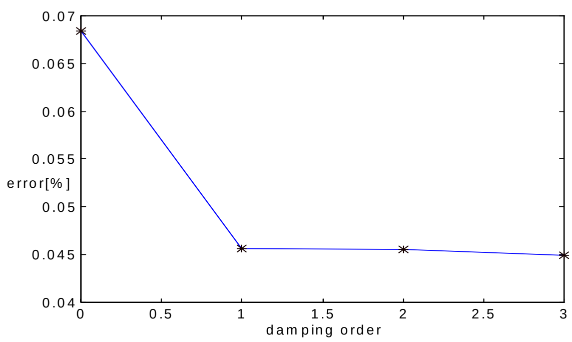
Case N°	Initial displacement cm	μ_0 (WT)	μ_0 (DPE)	μ_1 (WT) *Ns/m*	μ_1 (DPE) *Ns/m*	K (WT) *N/m*	K (DPE) *N/m*
1.	-2.5	0.13	0.12	0.28	0.27	13104	13099
2.	-2	0.12	0.12	0.28	0.27	13118	13131
3.	-1.5	0.12	0.12	0.26	0.24	13121	13126
4.	-0.5	0.11	0.11	0.29	0.25	13184	13206
5.	2	0.12	0.12	0.28	0.29	13113	13127
6.	2.5	0.14	0.14	0.28	0.30	13091	13073
7.	not measured	0.13	0.13	0.29	0.30	13107	13080
8.	not measured	0.14	0.13	0.27	0.30	13090	13093
9.	not measured	0.14	0.13	0.26	0.26	13110	13096

Table 3.1 : Identified parameters of the Low dissipation system

The mean squared errors versus the damping order is presented in Figure 3.1 (a) and (b) (respectively for W and DPE methods). In the case of the WT method, the minimum of the function corresponds to a damping order equal to 1. For the DPE method, the mean squared error in Figure 3.1 (b) decreases first rapidly and then very slowly when the correct damping order is reached.



W (a)



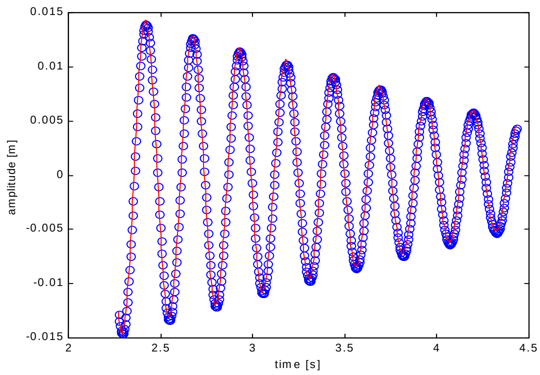
DPE (b)

Figure 3.1 : Error versus damping order (W and DPE)

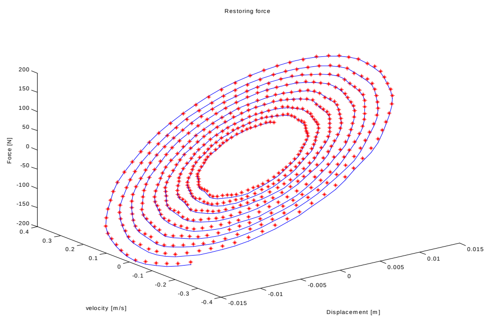
The comparison between the original and the reconstructed signals is given in Figure 3.2 (a), while in Figure 3.2 (b), the measured term $-m \cdot \ddot{u}$ is compared to the restoring force $f(u, \dot{u})$ (equation (2.13)) reconstructed from the identified parameters, that is according to the following expression:

$$f(u, \dot{u}) = m \cdot \mu_0 \cdot sign(\dot{u}) + m \cdot \mu_1 \cdot \dot{u} + K \cdot u \quad (3.1)$$

Note that the velocity used in equation (3.1) was obtained by the integration of the acceleration.



(a) Original (-) and reconstructed signal (o).



(b) Comparison between measured and reconstructed restoring force

Figure 3.2 : displacement and restoring force

Comparison of results

The results obtained by the DPE method and the WT technique are compared in Table 3. . Note that Table 3. gives the average values and the corresponding standard deviations for all the results obtained in the different test configurations.

	K N/m	σ_K N/m	μ_0	σ_{μ_0}	μ_1 Ns/m	σ_{μ_1} Ns/m
WT	13115.3	28	0.13	0.01	0.277	0.01
DPE	13114.6	40	0.12	0.01	0.276	0.02

Table 3.2 : Identified parameters : comparison of results

4. CONCLUSIONS

The comparison of results obtained from two different identification methods (the WT technique and the DPE method) on the same experimental SDOF system has shown a good agreement. The WT technique has the advantage that it requires to measure only the displacement of the system, while the DPE method needs the knowledge of the acceleration, velocity and displacement; for this reason, a lot of care must be dedicated in assuring and checking the perfect simultaneity between the signals. In most practical cases, it is a particular hard task to perform, since the sensors and particularly their amplification devices, often introduce very small time lags between the two recorded responses. In our case, we experienced a small phase difference and hence, in a first stage, the results obtained by the DPE method did not match the results obtained by the WT technique. For this reason, it was decided to calculate the displacement data by double integration of the acceleration signal. This procedure led to the results presented in Table 3., but a great effort was spent in processing the data and in optimising the integration routines.

On the other hand, the WT technique always requires a theoretical model as a baseline for the identification, while the DPE method allows to consider all possible forms of non-linearity, and hence represents a more general identification tool.

In the future, some supplementary tests should be performed on more complicated structures to extend and compare the different methods on multiple degrees of freedom systems.

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